Math 4200-001
Wednesday September 2: Finish 1.3; begin 1.5, complex differentiability

Announcements
Today we will add to the discussion of section 1.3, using Monday's notes. There is more to say about exponentials and logarithms and the complex "trig" functions. Then we will proceed into today's notes.

Notice that we're skipping section 1.4. This section is a review of some of the analysis you've learned in Math 3210-3220 in the context of the complex plane. Since the complex plane $\mathbb{C}$ is *isometric* to $\mathbb{R}^2$, I'm choosing to not formally cover section 1.4. Rather, we will use the definitions and theorems as we need them in 1.5 and going forward, and we will remind each other of how they correspond to - or actually exactly are - definitions and theorems from 3210-3220. We'll refer back to section 1.4 as needed. This is a change from how I taught the class last fall, and I think it will be more efficient and improve the flow. We'll see. :-)

Quiz 2 today, at the end of class, related to section 1.3 material.

Warm-up exercise

Def Let $f: A \to \mathbb{C}$ where $A \subseteq \mathbb{C}$ is *open*. Let $z_0 \in A$. We say that f *is (complex) differentiable at* z_0 iff

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} := f'(z_0)$$

exists. Note: an equivalent way to express the limit above is as

$$\lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h}.$$

Example 1: Using $|z - w|$ for the Euclidean distance between $z, w \in \mathbb{C}$, write down the precise statement of each analysis concept in the definition above for f being complex differentiable at z_0.

Def Let $f: A \to \mathbb{C}$ where $A \subseteq \mathbb{C}$ is open. If f is complex differentiable for all $z \in A$ then we say that f is *(complex) analytic* or *holomorphic* on A.

Remark: So that you don't get complacent, here is some magic we'll be seeing within a few weeks:

(i) If f is analytic on A as on the previous page, then the derivative function f' is too! And $f'' := (f')'$ is too. And in fact, f has n^{th} order derivatives of every order n on A as soon as its first derivative exists on all of A. Automatically! (Nothing like this was true in general for differentiable functions in regular Calculus! For example there are lots of differentiable functions that are not infinitely differentiable.)

(ii) If f is analytic on all of $\mathbb{C}$ and if f is also a bounded function, then actually f must be a constant. (This is called Liouville's Theorem.) In fact, if f is analytic on all of $\mathbb{C}$ and if f grows no faster than a polynomial ($|f(z)| \leq C|z|^n$ for $|z| \geq M$ some M), then actually $f(z)$ *is* a polynomial of degree at most n !! There are lots more analytic functions than just polynomials, but even if they're analytic on all of $\mathbb{C}$ they behave much more wildly than polynomials as $|z| \to \infty$.

(iii) If f, g are both analytic on an *open connected* set A and if $\{z_n\}_{n \in N} \subseteq A$ is a sequence of distinct complex numbers, with $\{z_n\} \to z_0 \in A$; and if $f(z_n) = g(z_n)$, $\forall\, n \in \mathbb{N}$, then actually $f(z) = g(z)\ \forall\, z \in A$!!!

Example 2 What are the two equivalent definitions of *connected set*, in the case that the set A is also *open*?

(iv) If f, g are both analytic on and open connected set A and if the function values and all derivatives of f and g agree at z_0 then actually $f(z) = g(z)$ for all $z \in A$.

Until we get to the magic, let's proceed as we did in Calculus. As we do this we'll be recalling facts and limit theorems/estimates from 3210-3220.

<u>Theorem</u> Let f be complex differentiable at $z_0 \in A$, $A \subseteq \mathbb{C}$ open. Then f is *continuous* at z_0.

<u>Theorem</u> Let $A \subseteq \mathbb{C}$ open, $f, g : A \to \mathbb{C}$ analytic, $c \in \mathbb{C}$. Then $c f$, $f + g, f g$ are analytic on A. And the quotient $\frac{f}{g}$ is analytic in A intersect the complement of the zero set for g. Furthermore, for $z \in A$,

(i) $(c f)'(z) = c f'(z)$

(ii) $(f + g)'(z) = f'(z) + g'(z)$

(iii) $(f g)'(z) = f'(z)g(z) + f(z)g'(z)$

(iv) $\left(\frac{f}{g}\right)'(z) = \frac{f'(z)g(z) - f(z)g'(z)}{g(z)^2}$ where $g(z) \neq 0$.

The proofs are just like in Calc 1. We can verify the product rule or the quotient rule, for example:

Some more computations that go just like in Calculus:

(i) if $f(z)$ is the constant function, its derivative is zero.

(ii) if $f(z) = z^n$, $n \in \mathbb{N}$, then $f'(z) = n\, z^{n-1}$

(iii) if $f(z) = z^n$, $n \in \mathbb{Z}$, then $f'(z) = n\, z^{n-1}$

(iv) every polynomial in z is analytic on $\mathbb{C}$, with the expected formula for its derivative.

(iv) if $f(z) = \dfrac{p(z)}{q(z)}$ is a rational function, i.e. a quotient of two polynomials, then $f(z)$ is analytic on the complement of the zero set for q.

The chain rule is also true - we'll prove this on Friday or next week, along with a discussion of the inverse function theorem. (The chain rule proof proceeds just like the precise proof for the 1-variable real chain rule that you discussed in 3210). In any case, if f is differentiable at z_0 and g is differentiable at $f(z_0)$ then $g \circ f$ is differentiable at z_0, and

$$(g \circ f)'(z_0) = g'(f(z_0))\, f'(z_0).$$

Example 3: Write $z = x + i\,y,\ y \in \mathbb{R}$. Then $f(z) = \mathrm{Re}(z) = x$ is NOT complex differentiable at any point of $\mathbb{C}$! (Even though the associated $F : \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$F(x, y) = (\mathrm{Re}\, f,\ \mathrm{Im}\, f) = (x, 0)$$

is Math 3220-differentiable, with differential (Jacobian) matrix

$$dF_{(x,\,y)} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} !!!$$

The way to check Example 3 at any point $z_0 = x_0 + i\,y_0$ is to evaluate the limits

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

$z \to z_0$ from the real and imaginary directions and see that these limits do not agree.

In fact, being complex differentiable is very rare for a function $f : A \subseteq \mathbb{C} \rightarrow \mathbb{C}$, relatively speaking, even when $\mathrm{Re}(f)$ and $\mathrm{Im}(f)$ are nice real-differentiable functions of x and y

<u>Theorem</u> Let $A \subseteq \mathbb{C}$ open, $f : A \rightarrow \mathbb{C}$, $z_0 \in A$. Write $f(z) = f(x + i\,y) = u(x, y) + i\,v(x, y)$, where $u(x, y) = \mathrm{Re}(f(x + i\,y)$, $v(x, y) = \mathrm{Im}(f(x + i\,y)$. Then if f is complex differentiable at $z_0 = x_0 + i\,y_0$ the following partial derivative equalities - known as the *Cauchy-Riemann equations* - must hold there:

$$u_x(x_0, y_0) = v_y(x_0, y_0)$$
$$u_y(x_0, y_0) = -v_x(x_0, y_0).$$

(The converse statement is almost true. The precise fact, which we'll discuss on Friday, is that if $F : A \subseteq \mathbb{R}^2 \rightarrow \mathbb{R}^2$, $F(x, y) = (u(x, y), v(x, y))$ is *Real differentiable* at (x_0, y_0) as you discussed in Math 3220, and if the CR equations hold at (x_0, y_0), *then* $f(x + i\,y) = u(x, y) + i\,v(x, y)$ is complex differentiable at $z_0 = x_0 + i\,y_0$. This is Theorem 1.5.8 in the text, which calls it the "Cauchy-Riemann Theorem". Geometrically, the CR Equations are saying that the differential map of F is given by a rotation-dilation matrix.)

proof: